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On Simulation of Oscillations of Given Form

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A class of periodic processes has been researched for which a self-oscillator can be synthesized with one degree of freedom by providing the periodic mode stability as a whole.

Key words: periodic process, oscillation form, operator with one degree of freedom

The results of studying the mathematical model of generators of the second order in the form of Van der Pole equation indicate that when ε is changed the form of oscillations generated is changed over a rather wide range, from sinusoidal up to relaxation [1] which form is close to rectangular. Therefore a problem regularly arises of constructing the generator of the second order for any given form of curve.

In the works dedicated to the synthesis of generators of oscillations of a given form it was shown that when only systems of the second order were used both theoretical and practical difficulties arise [2]. Using the method proposed in [2], the oscillations of an arbitrary form may be synthesized on the basis of the second order system using certain limitations. But the issue remains open of stability of the periodic mode to be constructed, namely, of the limiting cycle corresponding to this mode.

A simple mathematical model is known where the generator of strictly harmonic oscillations is realized with a single totally stable limiting cycle:

$$\frac{du}{dt}=-v+\varepsilon\left(1-u^2-v^2\right)u;\quad \frac{dv}{dt}=u+\varepsilon\left(1-u^2-v^2\right)v. \tag{1}$$

The periodic solution $u=\cos t, v=\sin t$ corresponds to this limiting cycle. By [2] the variables are changed in (1):

$$u=\varphi_1(x,y);\quad v=\varphi_2(x,y), \tag{2}$$

where $\varphi_1(x,y)$, $\varphi_2(x,y)$ are the single-valued differentiated functions. Then the differential equations relative to $x(t), y(t)$ are

$$\frac{dx}{dt}=\frac{1}{J}-\frac{1}{2}\frac{\partial}{\partial y}\left(\varphi_1^2+\varphi_2^2\right)+\varepsilon\left(1-\varphi_1^2\varphi_2^2\right)\left(\varphi_1\frac{\partial\varphi_2}{\partial y}-\varphi_2\frac{\partial\varphi_1}{\partial y}\right);$$

$$\frac{dy}{dt}=\frac{1}{J}\frac{1}{2}\frac{\partial}{\partial x}\left(\varphi_1^2+\varphi_2^2\right)-\varepsilon\left(1-\varphi_1^2-\varphi_2^2\right)\left(\varphi_1\frac{\partial\varphi_2}{\partial x}-\varphi_2\frac{\partial\varphi_1}{\partial y}\right), \tag{3}$$

where

$$J=\begin{vmatrix}\frac{\partial\varphi_1}{\partial x} & \frac{\partial\varphi_1}{\partial y}\\ \frac{\partial\varphi_2}{\partial x} & \frac{\partial\varphi_2}{\partial y}\end{vmatrix}.$$

For the system (3) as well as (1) to have a solution of Cauchy problem with all values $x(0), y(0)$ it is necessary and sufficiently to meet the conditions [3]:

$$J\neq 0\ \forall (x,y)\in R^2;\quad \lim_{(x^2+y^2)\to\infty}\left(\varphi_1^2+\varphi_2^2\right)=\infty.$$

In this case (2) is a mutual single-valued transform: for all $x,y\in R^2$ the single-valued functions $x=F_1(u,v),\ y=F_2(u,v)\ \forall (u,v)\in R^2$ exist.

The topological structures on the planes x, y, u, v coincide. This means that the limiting cycle on the plane x, y is stable as a whole and consequently, the periodic modes $\tilde{x}(t-t_0), \tilde{y}(t-t_0)$ are also stable. Here t_0 is the arbitrary constant. But it is unknown whether this transform exists for the oscillation $\tilde{x}(t)$ of arbitrary form. In the case when Jacobian does not change the sign on the circumference $u^2+v^2=1$ and in its certain neighbourhood the required periodic mode may be constructed. However its stability as a whole cannot be guaranteed.

The construction of generator of oscillations of arbitrary form $\tilde{x}(t)$ is reduced to the synthesis of $F_1(u,v)$ function which has to meet the condition:

$$\tilde{x}(t)=F_1(\cos t,\sin t), \tag{4}$$

and the choice $F_2(u,v)$ is bounded by the requirement

$$I=\begin{vmatrix} \dfrac{\partial F_1}{\partial u} & \dfrac{\partial F_1}{\partial v} \\ \dfrac{\partial F_2}{\partial u} & \dfrac{\partial F_2}{\partial v} \end{vmatrix} \neq 0 \;\; \forall\, (u,v)\in R^2, \quad \lim_{(u^2+v^2)\to\infty}\left(F_1^2+F_2^2\right)=\infty. \tag{5}$$

The necessary condition which the function $F_2(u,v)$ must satisfy is that the curve Γ: $\tilde{x}(t)=F_1(\cos t,\sin t)$, $y(t)=F_2(\cos t,\sin t)$ representing the limiting cycle on the plane, should be a simple curve, i.e. It should not have self-intersections. To construct the curve a limitation is imposed on $\tilde{x}(t)$: let $\max\limits_{0\le t\le 2\pi} \tilde{x}(t)=x_m$, then the highest local maximum $x_l<x_m$ is assumed. Without generality limitation $x(0)=x_m$ is taken, then

$$y(t)=\begin{cases} \dfrac{t}{T}, & 0\le t\le T-\tau, \\ \dfrac{T-\tau}{\tau}\left(1-\dfrac{t}{T}\right), & T-\tau\le t\le T, \end{cases} \quad T=2\pi. \tag{6}$$

If τ is chosen in such a way that $\tilde{x}(T-\tau)>x_l$, the curve Γ has no self-intersections.

The above method of $y(t)$ construction reproduces the scanning on the oscilloscope screen and certainly is not a unique one. The condition (4) may be satisfied by at least two methods. The first is in representation $F_1(u,v)$ in the form of linear combination of Chebyshev polynomials of the first and second kind. If

$$F_1(u,v)=a_0+\sum_{k=1}^{\infty}\left[a_k\,T_k(u)+b_k\,v\,U_{k-1}(u)\right],$$

is taken where

$$T_k(u)=\frac{k}{2}\sum_{j=0}^{[k/2]}(-1)^j\,\frac{(k-j-1)!}{j!\,(k-2j)!}(2u)^{k-2j};$$

$$U_k(u)=\sum_{j=0}^{[k/2]}(-1)^j\,\frac{(k-j)!}{j!\,(k-2j)!}(2u)^{k-2j},$$

then assuming $u(t) = \cos t$, $v(t) = \sin t$ we obtain

$$\tilde{x}(t) = \sum_{k=0}^{\infty} (a_k \cos kt + b_k \sin kt). \tag{7}$$

$F_2(u, v)$ may be similarly obtained in the form of linear combination of Chebyshev polynomials if $y(t)$ from (6) is represented as Fourier series. Then it is valid that the topological mapping of the circumference $u^2 + v^2 = 1$ onto the curve Γ is obtained. But we cannot yet state that this mapping is regular on the circumference and in the circle $u^2 + v^2 \le 1$, i.e. the question remains open for which domain the Jacobian of transform is not vanished [4].

The second method is based on the Riemann theorem by which the schlicht function $w = f(z)$, $z = x + jy$, $w = u + jv$ exists which maps the domain D limited by the curve Γ onto the interior of the single circle of the plane w and satisfying the conditions $f(z_0) = 0$, $f'(z_0) > 0$ at the given point z_0 of the domain D [5].

The Jacobian of transform $u = \varphi_1(x, y)$, $v = \varphi_2(x, y)$ in the case of analytical function $f(z)$ is

$$J = \left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2 = |f'(z)|^2 \ge 0.$$

Since the function is schlicht in D, then $f'(z) \ne 0$, $\mathrm{V} - z \leftrightarrow D$. Thus the difference of Jacobian from zero inside the curve Γ is guaranteed by the Riemann theorem. But it is unknown whether the derivative $f'(z)$ differs from zero on the contour Γ and how close to the contour Γ zeros $f'(z)$ are located outside it.

These considerations are not constructive since it is well known [5] that the construction of the function $f(z)$ for a contour of a complex form is extremely difficult and the form of the functions $\varphi_1(x, y)$, $\varphi_2(x, y)$ is very complex. Besides, the mapping on the basis of the Riemann theorem makes it possible to map the circumference in the contour Γ of an arbitrary form. However this does not mean the form of the function $x(t)$ coincides with the given one. By the Riemann theorem the coincidence with the given curve $\tilde{x}(t)$ can be reached for not more than three points. Therefore the form of functions which may be realized is rather limited. Really, if $f(z)$ is represented as the Taylor series

$$w = \sum_{k=0}^{\infty} \alpha_k z^k, \tag{8}$$

where $\alpha_k = A_k e^{j\Psi_k}$ and if $z = e^{jt}$ is assumed, i.e. $u = \cos t$, $v = \sin t$, we obtain

$$w = \sum_{k=0}^{\infty} A_k e^{j(kt + \Psi_k)},$$

or

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(kt + \Psi_k), \quad y(t) = \sum_{k=1}^{\infty} A_k \sin(kt + \Psi_k).$$

The transform (7) would seem to permit us to obtain the oscillations of arbitrary form. Note that $y(t) = \sum_{k=1}^{\infty} A_k \cos(kt + \Psi_k - \pi/2)$ is the Hilbert transform of $x(t)$ function. But the univalence condition implies significant limitations on the class of the functions to be synthesized. If $f(z)$ is represented as the Taylor series (8) then despite the absence of necessary and sufficient conditions of univalence, the necessary conditions $|\alpha_2| \le 2$, $|\alpha_3| \le 3$ and sufficient condition $\sum_{n=2}^{\infty} n|\alpha_n| \le 1$ may be used.

From the necessary univalence condition it follows that the amplitudes of the second and third harmonics cannot exceed that of the first harmonic two and three times respectively. In fact these necessary conditions are far from sufficient. For $f(z) = z + \alpha_2 z^2$, the necessary and sufficient univalence condition $|\alpha_2| < 1/2$, i.e. the amplitude of the second harmonic in the signal to be synthesized must not exceed half of the amplitude of the main frequency. In this case the sufficient condition coincides with the necessary one. From the sufficient condition it follows that the univalence is provided if $|\alpha_n| \le 1/n^3$. This means that the signal synthesis by the transform $w = f(z)$ is possible for the cubic law of harmonics decrease depending on their number.

The above considerations indicate that the use of analytical functions for mapping $(u, v) \to (x, y)$ significantly restricts the class of functions which can be synthesized with generator with one degree of freedom. Therefore for estimating the possibilities of systems of the second order two methods may be appropriate.

1. Somewhat to restrict the class of functions to be generated with saving global stability.

2. To reject the demand of global stability and to try to construct generator which synthesises the oscillations of arbitrary form.

Very wide class of oscillations may be obtained if

$$x = F(u) + g(u)\,h(v); \quad y = -u + x, \tag{9}$$

where $F(u)$ is the linear combination of the Chebyshev polynomials which provides in $x(t)$ the cosine series with arbitrarily given coefficients of harmonics with $u = \cos t$ to be substituted. The second summand in the expression for x is necessary to obtain sine components of Fourier series.

The transform Jacobian (9) is $J = g(u)\,h'(v)$ and for the condition $J > 0$ to be met $g(u) > 0 \ \forall\, u \in R,\ h'(v) > 0 \ \forall\, v \in R$ should be assumed. It is known that the Jacobian inequality to zero does not yet provide one-to-one mapping. The uniqueness of the inverse mapping should be corrected. This is possible in our case. When the relationships for x and y (9) are added and subtracted we obtain

$$u = x - y; \quad h(v) = \frac{x - F(x-y)}{g(x-y)}.$$

The uniqueness of the inverse mapping $\forall\,(x, y) \in R^2$ follows from the fact that the function $g(u) = c,\ c << 1$ is monotonous.

Thus the above mapping makes it possible to obtain any predetermined amplitude spectrum if only cosine components of Fourier series are taken into account. It is sufficient here to assume $g(u) = c,\ c << 1$. The given phase oscillation spectrum is somewhat more difficult to be obtained.

Consider a special case

$$g(u) = 1 + a_1 u + a_2 u^2; \quad h(v) = \sum_{k=1}^{\infty} b_k \frac{v^{2k-1}}{2k-1}.$$

For $g(u)$ to be positive the inequality should be satisfied

$$a_2 > \frac{a_1^2}{4} \tag{10}$$

and the coefficients of the series for $h(v)$ must be those that

$$h'(v) = \sum_{k=1}^{\infty} b_k\, v^{2(k-1)} \neq 0 \quad \forall\, v \in R.$$

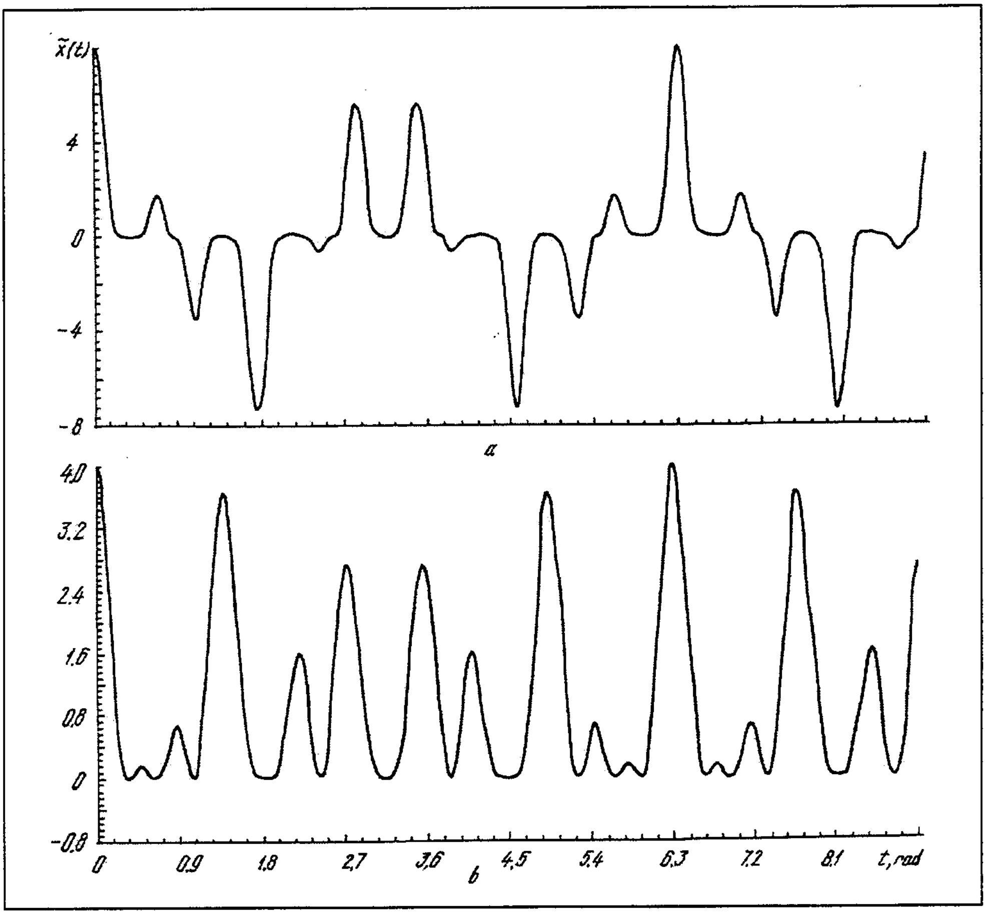

Figure 1. Periodic oscillations $\tilde{x}$ realized by self-oscillator: a $-\tilde{x}(t) = (\cos(2t) + \cos(9t))^3 + 0.04 \sin t$; b $-$ $\tilde{x}(t) = (\cos(2t) + \cos(7t))^2 + 0.04 \sin t$

The series $h(v)$ involves only odd powers v in order not to generate additional cosine summands in $x(t)$ which were already obtained by substituting $u = \cos t$ in $F(u)$. If $h(v) = v$ is assumed, then

$$g(\cos t)\sin t = \left(1 + \frac{1}{4}a_2\right)\sin t + \frac{a_1}{2}\sin 2t + \frac{a_2}{4}\sin 3t. \tag{11}$$

The relationship (10) shows that the coefficients in sine sum cannot be arbitrarily chosen: the third harmonic cannot exceed the value of the first one, and

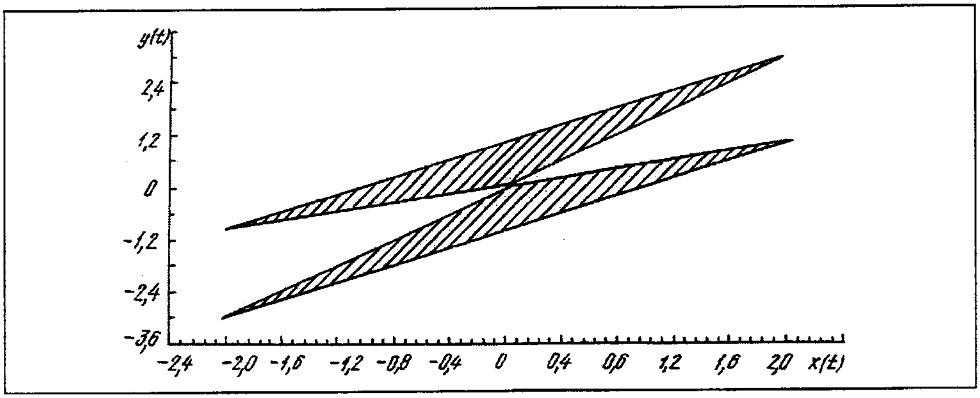

Figure 2. Phase portrait of oscillations of the second order

the second harmonic with $a_1 = 1$ should be 2.5 lower than the first one. The limitations on the amplitudes of higher harmonics of sine components is also preserved. We shall try to show how they influence the forms of the curves which may be realized.

The Fourier series with cosine components describes the even functions or more precisely, the functions which are symmetric relative to extremal points (if it is taken into account that the substitution $u = \cos(t - t_0)$ is possible). Arbitrarily great asymmetry may be obtained if the sine component of the main frequency is introduced in $x(t)$ using the second summand. Then the limitations on the oscillations to be realized are insignificant: this is the difficulty of creating arbitrary asymmetry relative to extremal point by high frequency harmonics. The possibility of realization of a wide class of oscillations may be seen from the plots of functions to be synthesized shown in Fig. 1.

The second way of expanding the class of the functions generated may be also studied on the mapping (9) if the conditions of global stability of periodic solutions are rejected. For example, if the condition (10) is not fulfilled, the condition (11) is violated on straight lines in the plane $u = 05\, a_2 \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right)$. In the plane (x, y) it is equivalent to the condition

$$y = x - 05\, a_2 \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right). \qquad (12)$$

For the self-oscillating system with the given curve form to be realized, the straight lines (12) must not intersect the boundary cycle Γ. Numerical calculations indicate that significant expansion of the class of realized curves is not observed. Other possible ways are not noteworthy.

The possibility of creating self-oscillating system with one degree of freedom where oscillations may have a rather complex form, permits another point of view as to the qualitative difference between systems of the second and higher order. For example, Fig. 2 presents the phase portrait

$$\tilde{x} = 0.1\sin t + \cos 197t + \cos 199t; \quad y = -\cos t + \tilde{x},$$

obtained for a generator of the second order. The existence of two harmonic oscillations with the frequency ratio 199/197 makes them practically indistinguishable from quasi-periodic oscillations with two basis frequencies.

The frequency ratio p / q may be in principle obtained for any high values p, q. The increase of "basis" frequencies generated by the system of oscillations of the second order also does not offer many difficulties.

Note that two black triangles in Fig. 2 with high resolution depict a closed simple curve without self-intersections. The difference between systems of the second and higher order is reduced to the fact that in the first case the frequency relationship of separate oscillations is the rational number and in the second case it is irrational. But with high mutually simple p, q this difference is not discovered from the point of view of the experimenter because" the physician never encounters irrational numbers" [6].

Further complications of oscillations are possible in systems of the second order when they become practically indistinguishable from chaotic oscillations. In any case, if we try to classify these oscillations by fractional dimensionality on the basis of Takens results [7] or by the spectrum form, the results of complex periodic oscillation studies appear indistinguishable from those of chaotic oscillations. It is clear that this conclusion is not original but attention was probably not given to its validity for systems with one degree of freedom.

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